Fate and Transport of Pollutants in Lake Systems

Lakes and human-made reservoirs serve as valuable drinking water resources. While many small lakes remain pristine, most human-made lakes suffer from over-development and large lakes are subject to contamination from local industrial sources and shipping accidents. Regardless of the size of the lake, most introductory modeling efforts simplify the governing equations by assuming that the lake is completely mixed immediately after the addition of a contaminant. It is also assumed that the volume of the lake does not change over the time interval of study, so that the volume of water entering the lake is equal to the volume of water exiting the lake, usually in the form of a stream.

Conceptual Development of Governing Fate and Transport Equation

Instantaneous Pollutant Input: Before we show the mathematical development of the governing equation, we will present a conceptual approach that shows how each part of the equation relates to a physical model of the lake (Figure 1). Two views of the lake are shown in this figure. The first figure shows a bird's eye view of the lake with the water entering the lake on the left and exiting on the right. The governing equation is shown in the center of the figure. The concentration of pollutant in the exiting water is shown as a function of time elapsed since input, in the upper right-hand corner of Figure 1. The bottom figure shows a cross-section of the lake.

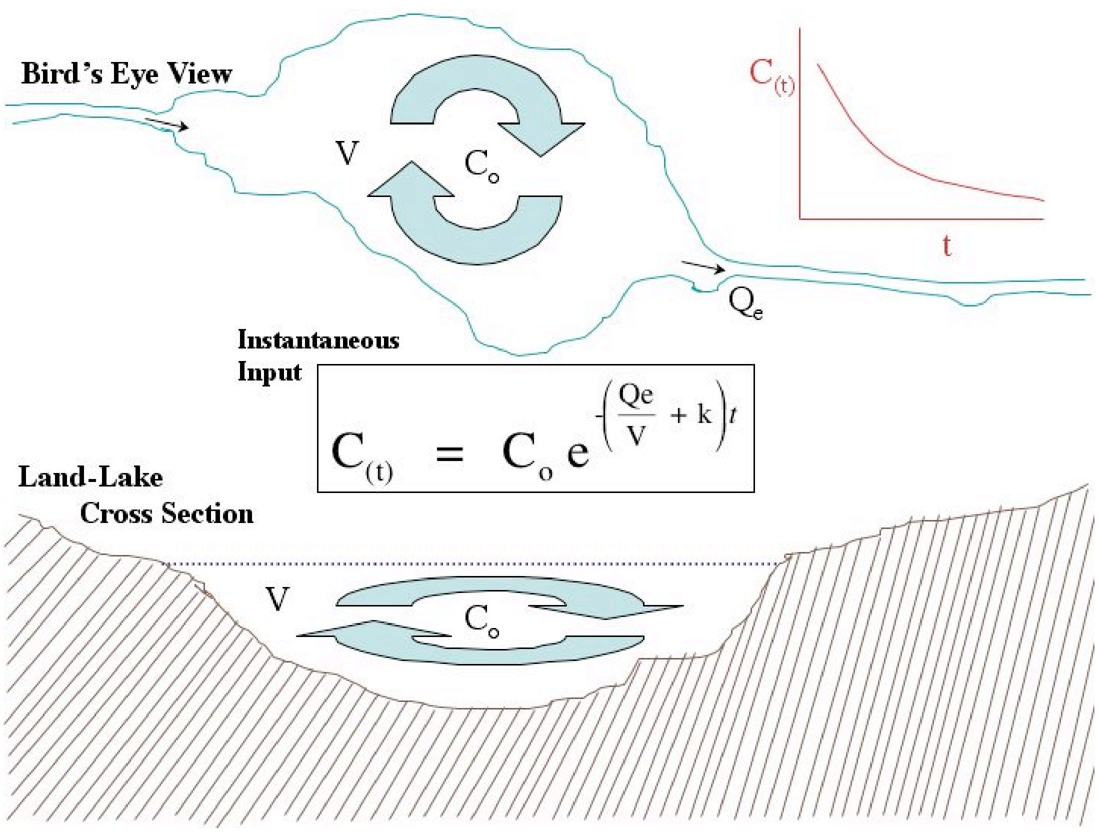

Figure 1. Pollutant Concentrations in a lake following an instantaneous input.

First we assume that the input of pollutant is evenly distributed over the entire lake and that the lake is completely mixed. Thus, the total mass of pollutant added to the lake is divided by the volume of the lake to yield the initial pollutant concentration, C_o. Next we will look at how pollution is removed from the lake. Our model assumes that there are two ways of removing pollution from the lake: degradation (either microbial, chemical, sorption, or volatilization) described by the first-order rate constant (k) in the governing equation, and natural out of the lake with the river water (represented by Q_e). Since the lake is completely mixed and the pollutant concentration is equal everywhere in the lake, the concentration of pollutant in the exiting river is the same as the concentration in the lake. This concentration is represented by C_t in the governing equation and is the concentration at a specific time after the addition of pollutant to the lake. As time passes (t increases) the concentration of the pollutant in the lake and in the exiting water can be calculated using the equation for an instantaneous pollutant input. This accounts for all of the terms in the governing equation. A more mathematical approach to our modeling effort is described later in this section.

Mathematical Approach to a Lake System

The first step in developing the governing equations for the fate of a pollutant in a lake system is to set up a mass balance on the system. First, quantify all of the mass inputs of pollutant to the system. This can be expressed as

$$W = Q_w C_w + Q_i C_i + Q_{trib} C_{trib} + P A_s C_p + V C_s$$

where W = the mass input of pollutant rate per unit time, kg/time

- Q_w = the inflow rate of the waste water, m^3/time
- C_w = the pollutant concentration in the waste water, kg/m^3
- Q_i = the inflow rate of the main river, m^3/time
- C_i = the pollutant concentration in the main inlet river, kg/m^3
- Q_{trib} = the net inflow rate from all other tributaries, m^3/time
- C_{trib} = the net pollutant concentration of in the tributaries, kg/m^3
- P = annual precipitation, m/time
- A_s = mean lake surface area, m^2
- C_p = the net pollutant concentration in precipitation, kg/m^3
- V = the average lake volume, m^3
- Cs = the average pollutant release from suspended lake sediments, kg/m^3-time

In most situations, the mass inputs from the smaller tributaries and precipitation are minor compared to the major input source and these terms are ignored. We will further simplify the mass input expression here by assuming that the contribution from contaminated sediments is negligible, but this is not always the case. These assumptions simplify the input expression to

$$W = Q_w C_w + Q_i C_i$$

Next, we set up a mass balance for the pollutant across the entire system,

$$\text{Change in mass} = \text{Inflow} - \text{Outflow} + \text{Sources} - \text{Sinks}$$
$$VdC = (Q_w C_w\, dt + Q_i C_i\, dt) - Q_e C\, dt + 0 - VCk\, dt, \quad \text{or}$$
$$VdC = W\, dt - Q_e C\, dt - VCk\, dt$$

where
dC = the change in pollutant concentration in the lake,
dt = the incremental change in time,
Q_e = the outlet or effluent flow from the lake
C = the average lake concentration, kg/m^3 and
k = the first-order removal rate for the pollutant, 1/time.

The last equation, upon rearrangement, yields

$$Q_e C - W(t) + dVC = -VCk\, dt$$

and if the Q_e, k and V of the lake are assumed to be constant, this equation, upon rearrangement, reduces to

$$V\frac{dC}{dt} + (Q_e + kV)C = W(t)$$

If the average detention time (t_o) of the water (and thus the pollutant) in the lake is defined as

$$t_o = \frac{V}{Q}$$

Substitution and further rearrangement into the previous equation, yields

$$V\frac{dC}{dt} + CV\left(\frac{1}{t_o} + k\right) = W \qquad \text{Eqn 1}$$

This is a first-order linear differential equation.

Instantaneous Pollutant Input Model

When the mass input from all sources, W(t), is zero, we approach what is referred to as an instantaneous input. In this case, an instantaneous input is characterized as a one-time, finite addition of pollutant to the lake. For example, the release of a pollutant by a marine shipping accident would be an instantaneous input, as would a short release from an industry located on the lake. Under these conditions, integration of Eqn 1 with W = 0, yields

$$C(t) = C_o\, e^{-\left(\frac{Qe}{V} + k\right)}, \text{ or}$$

$$C(t) = C_o\, e^{-\left(\frac{1}{to} + k\right)}$$

The second equation above would be used to simulate the pollutant concentration in a lake where an instantaneous release occurred.

References:

Metcalf & Eddy, Inc., 1972. Wastewater Engineering: Collection, Treatment, Disposal. McGraw Hill Series in Water Resources and Environmental Engineering, McGraw-Hill, New York.

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