Explanation of Example Problem for an Instantaneous Input into a Lake

Here, we will use the same problem used in the step input example problem, but we will monitor the removal of the insecticide if the input is ceased after 1 year. We found in the previous problem that after 1 year the average concentration in the lake was 12.2 mg/L. The previous problem statement was: "A lake in a rural community has an average surface area of 5 km^2 and a mean depth of 50 m. A stream exits the lake with an average annual flow rate of 45,000 m^3/yr. Aerial application of an insecticide in the area introduces the compound into the lake. The average annual loading to the lake from the atmospheric and from agricultural runoff is estimated at 50 kg/day. Assuming a first-order removal of the insecticide (half-life = 45 days) from the lake and the initial background concentration of insecticide in the lake are negligible:"

From the previous problem, the volume of the lake is equal to the average surface area times the mean depth,

$$\text{Volume} = (5000\ m^2)\ (\ 50\ m) = 250{,}000\ m^3$$

$$\text{Detention time} = \frac{250{,}000\ m^3}{45{,}000\ m^3/yr} = 5.56\ yr$$

In order to solve this problem, we must also convert the first-order half-life to a rate constant, k, expressed in units of reciprocal years. The half-life of 45 days is equal to a half-life of 0.12 years.

$$\ln \frac{C}{Co} = -\ kt$$

$$\ln (0.5) = -\ k\ (0.12\ yr)$$

$$k = 5.78 / yr$$

From the previous problem, the concentration after 1 year is

$$C = \frac{W}{\beta V}\left(1 - e^{-\beta t}\right)$$

$$\text{where } \beta = \frac{1}{t_o} + k$$

$$\square = \frac{1}{t_o} + k = \frac{1}{5.56} + 5.78 = 5.96$$

$$C = \frac{W}{\square V}\left(1 - e^{-\square t}\right)$$

$$C = \frac{(18250\ kg/yr)(1000g/kg)(10^3 mg/g)}{5.96(250000m^3)(1000L/m^3)}\left(1 - e^{-5.96*1}\right)$$

$$C = 12.25\left(1 - e^{-5.96*1}\right)$$

$$C = 12.25 - 12.25(2.58 \times 10^{-3})$$

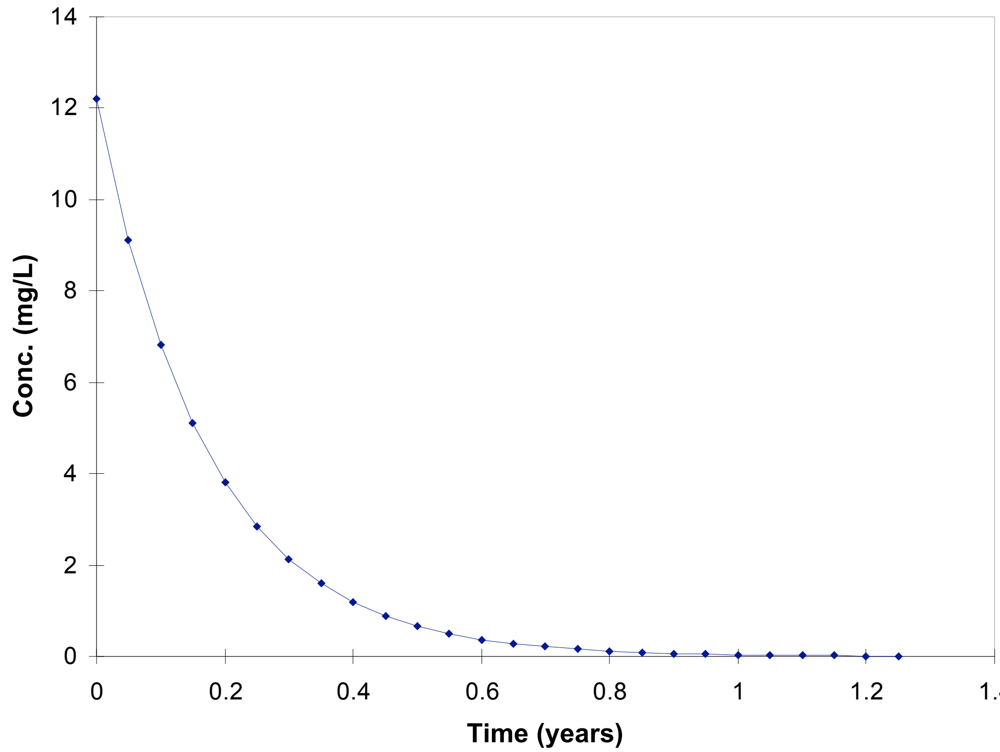
$$C = 12.2\ mg/L$$

(1) Now, for this problem calculate the insecticide concentration as a function of time. The equation governing this process is

$$C(t) = C_o\, e^{-\left(\frac{Qe}{V} + k\right)t}, \text{ or}$$

$$C(t) = C_o\, e^{-\left(\frac{1}{to} + k\right)t}$$

where Co = 12.2 mg/L. A plot of this equation is shown below.



(2) Calculate the time required to reach an insecticide concentration of 1000 mg/L, which happens to be the detection limit for this compound.

$$C(t) = C_o\, e^{-\left(\frac{Qe}{V} + k\right)t}, \text{ or}$$

$$C(t) = C_o\, e^{-\left(\frac{1}{to} + k\right)t}$$

$$\frac{C(t)}{C_o} = e^{-\left(\frac{1}{to} + k\right)t}$$

$$\frac{0.100 \text{ mg/L}}{12.2 \text{ mg/L}} = e^{-\left(\frac{1}{5.56} + 5.78\right)t}$$

$$\ln \frac{0.100 \text{ mg/L}}{12.2 \text{ mg/L}} = -\left(\frac{1}{5.56} + 5.78\right)t$$

$$4.80 = 5.96\, t$$

$$t = 0.81 \text{ yr}$$

Thus, it is seen that the lake will recover very rapidly after the input of the insecticide is halted.